

# Thermodynamics of Lovelock-Lifshitz Black Branes

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## Abstract

We investigate the thermodynamics of Lovelock-Lifshitz black branes. We begin by introducing the finite action of third order Lovelock gravity in the presence of a massive vector field for a flat boundary, and use it to compute the energy density of these black branes. Using the field equations, we find a conserved quantity along the  $r$  coordinate that relates the metric parameters at the horizon and at infinity. Remarkably, though the subleading large- $r$  behavior of Lovelock-Lifshitz black branes differs substantively from their Einsteinian Lifshitz counterparts, we find that the relationship between the energy density, temperature, and entropy density is unchanged from Einsteinian gravity. Using the first law of thermodynamics to obtain the relationship between entropy and temperature, we find that it too is the same as the Einsteinian case, apart from a constant of integration that depends on the Lovelock coefficients.

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## I. INTRODUCTION

Gauge/gravity duality presents a powerful new framework with which one may study strongly coupled gauge theories. Indeed, it has been conjectured that there is a duality between certain strongly coupled gauge theories and weakly coupled string theories, which means that both theories describe the same physics, but calculations may be easier in one theory than the other. More specifically, the AdS/CFT correspondence relates  $\mathcal{N} = 4$  supersymmetric  $SU(N)$  gauge theory to superstrings in 10 dimensions [1, 2]. One typical and fascinating aspect of gauge/gravity duality is the property of holography, which states that the amount of information contained in the boundary gauge theory is the same as the one contained in the bulk string theory. This holographic toolbox has since spread its usage to many branches of physics from hydrodynamic [3] to condensed matter systems [4]. However these studies have been concerned with relativistic theories with a conformal symmetry in the ultraviolet, which are described by asymptotically Anti-de Sitter (AdS) spacetimes. Applying gauge/gravity duality to condensed matter theories with anisotropic scaling symmetry,

$$t \rightarrow \lambda^z t, \quad \mathbf{x} \rightarrow \lambda \mathbf{x} \quad (1)$$

where  $z$  is the dynamical exponent, suggests that we can further extend holography to encompass nonrelativistic and Lifshitz field theories.

From a holographic point of view, this suggests the following (asymptotic) form for the spacetime metric

$$ds^2 = L^2 \left( -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\mathbf{x}^2 \right) \quad (2)$$

that obeys the scale invariance

$$t \rightarrow \lambda^z t, \quad r \rightarrow \lambda^{-1} r, \quad \mathbf{x} \rightarrow \lambda \mathbf{x}. \quad (3)$$

noted previously in a braneworld context [5]. A four-dimensional anisotropic scale invariant background using an action involving a two form and a three form field with a Chern-Simons coupling or a massive vector field can be engineered to yield solutions with this asymptotic behavior [6]. For these matter fields, lots of effort has been expended in extending this solution to the case of asymptotic Lifshitz solutions. One of the first analytic examples was reported in Ref. [7] for a sort of higher-dimensional dilaton gravity without restricting the value of the dynamical exponent  $z$ . An exact topological black hole solution that is

asymptotically Lifshitz with  $z = 2$  was obtained in [8]; further solutions with  $z = 4$  and with spherical topology were subsequently obtained [9]. In general, however, such asymptotic Lifshitz black holes must be obtained numerically [8–10]. Asymptotic Lifshitz solutions in the vacuum of higher-derivative gravity theories (with curvature-squared terms in the action) have been investigated [11]; the higher-curvature terms with suitable coupling constant play the role of the desired matter. Recently, we introduced some solutions – both analytically and numerically – which can be regarded as higher-curvature modifications from Lovelock gravity to those obtained from Einsteinian gravity coupled to matter [12].

The holography of gravity theories including higher powers of the curvature, particularly curvature-cubed interactions, have attracted increased attention [13, 14]. This is because, in the context of AdS/CFT correspondence, corrections from higher powers of the curvature must be considered on the gravity side of the correspondence in order to investigate CFTs with different values of their central charges. On the other side, the asymptotic Lifshitz black hole from a holographic point of view needs an extension of holography to encompass nonrelativistic field theories. Our aim here is to further develop the holographic dictionary for asymptotically Lifshitz spacetimes in Lovelock gravity by introducing the counterterm method for Lovelock-Lifshitz black branes [12] with flat boundary. Indeed, the generalization of holographic techniques to this new context may offer new insights into the nature of the relation between quantum gravity in asymptotically non-AdS spacetimes and the dual field theory. Here we generalize the counterterm method introduced in [15] for four-dimensional Einstein gravity to  $(n+1)$ -dimensional Lovelock gravity, which of course contains the higher-dimensional Einstein gravity and Gauss-Bonnet gravity too. The counterterm method for asymptotic AdS solutions of Lovelock gravity with flat boundary was introduced by us in [16]. We generalize it to include asymptotic Lifshitz solutions of Lovelock gravity.

We make use of these generalizations to investigate the thermodynamics of Lovelock-Lifshitz black branes [12]. Somewhat remarkably we find that the relationship between the energy density, temperature, and entropy density is unchanged from Einsteinian gravity. We make use of the first law of thermodynamics to obtain the relationship between entropy and temperature. We find this is also the same as the Einsteinian case, apart from a constant of integration that depends on the parameters in the Lovelock action.

The outline of our paper is as follows. We introduce the one dimensional Lagrangian of the Lovelock-Lifshitz black branes in Sec. II. We also write down the field equations and

present the constant  $\mathcal{C}_0$  which is preserved along the coordinate  $r$ . This constant relates the information at the horizon and at infinity. In Sec. III, we generalize the counterterm method introduced for the four-dimensional Einstein equation [15] to Lovelock gravity, and present the finite action for Lovelock-Lifshitz black branes. Section IV is devoted to the calculation of the constant  $\mathcal{C}_0$  at the horizon and at infinity. In Sec. V, we obtain the energy density of the Lovelock-Lifshitz black branes in terms of the temperature and entropy and find that it is the same in Einstein [17] and Lovelock gravities. We also consider the thermodynamics of the solutions. We finish our paper with some concluding remarks.

## II. FIELD EQUATIONS

The bulk action of third order Lovelock gravity [18] in the presence of an Abelian massive vector field  $A^\mu$  may be written as

$$\begin{aligned} I_{\text{bulk}} &= \frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} (\mathcal{L}_g + \mathcal{L}_m), \\ \mathcal{L}_g &= \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 - 2\Lambda, \\ \mathcal{L}_m &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \end{aligned} \quad (4)$$

where  $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ ,  $\Lambda$  is the cosmological constant,  $\alpha_2$  and  $\alpha_3$  are second and third order Lovelock coefficients,  $\mathcal{L}_1 = R$  is the Einstein-Hilbert Lagrangian,  $\mathcal{L}_2 = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2$  is the second order Lovelock (Gauss-Bonnet) Lagrangian, and

$$\begin{aligned} \mathcal{L}_3 &= R^3 + 2R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}_{\mu\nu} + 8R^{\mu\nu}_{\sigma\rho} R^{\sigma\kappa}_{\nu\tau} R^{\rho\tau}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\nu\rho} R^\rho_{\mu} \\ &\quad + 3RR^{\mu\nu\sigma\kappa} R_{\sigma\kappa\mu\nu} + 24R^{\mu\nu\sigma\kappa} R_{\sigma\mu} R_{\kappa\nu} + 16R^{\mu\nu} R_{\nu\sigma} R^\sigma_{\mu} - 12RR^{\mu\nu} R_{\mu\nu} \end{aligned} \quad (5)$$

is the third order Lovelock Lagrangian. We assume that the Gauss-Bonnet coefficient, which has the dimension of (length)<sup>2</sup>, is positive as in heterotic string theory [19]. In Lovelock gravity only terms with order less than  $[(n+1)/2]$  (where  $[x]$  is the integer part of  $x$ ) contribute to the field equations, the rest being total derivatives in the action. For third order Lovelock gravity we therefore consider  $(n+1)$ -dimensional spacetimes with  $n \geq 6$  (though in situations where we set  $\hat{\alpha}_3 = 0$  our solutions will be valid for  $n \geq 4$ ).

We write the spherically symmetric gauge field and metric of an  $(n+1)$ -dimensional

asymptotically Lifshitz static spacetime with zero curvature boundary as

$$A = qe^{H(r)}dt = q\frac{r^z}{l^z}h(r)dt, \quad (6)$$

$$ds^2 = -e^{2F(r)}dt^2 + e^{2G(r)}dr^2 + l^2e^{2R(r)}\sum_{i=1}^{n-1}(dx^i)^2 \quad (7)$$

$$= -\frac{r^{2z}}{l^{2z}}f(r)dt^2 + \frac{l^2dr^2}{r^2g(r)} + r^2\sum_{i=1}^{n-1}(dx^i)^2, \quad (8)$$

where the new metric functions are related to our previous notation [12] through the following relations

$$\begin{aligned} F(r) &= \frac{1}{2}\ln f(r) + z\ln \frac{r}{l}, \\ G(r) &= -\frac{1}{2}\ln g(r) - \ln \frac{r}{l}, \\ R(r) &= \ln \frac{r}{l}, \\ H(r) &= \ln h(r) + z\ln \frac{r}{l} \end{aligned} \quad (9)$$

and in Eq. (8) we have chosen coordinates by taking  $R(r) = \ln(r/l)$ .

Our previous study of Lovelock-Lifshitz black holes [12] showed that if

$$\begin{aligned} m^2 &= \frac{(n-1)z}{l^2}, \quad q^2 = \frac{2(z-1)L^4}{zl^4}, \\ \Lambda &= -\frac{[(z-1)^2 + n(z-2) + n^2]L^4 + n(n-1)(\hat{\alpha}_2l^2 - 2\hat{\alpha}_3)}{2l^6}, \end{aligned} \quad (10)$$

where  $L^4 = l^4 - 2\hat{\alpha}_2l^2 + 3\hat{\alpha}_3$ , then the action (4) supports solutions asymptotic to the Lifshitz solution [12]

$$ds^2 = -\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2dr^2}{r^2} + r^2\sum_{i=1}^{n-1}dx_i^2. \quad (11)$$

Since we are assuming spherical symmetry, we can reduce the action to one dimension and subsequently obtain the equations of motion. After integration by parts, the one dimensional Lagrangian may be written as  $\mathcal{L}_{1D} = l^{n-1}(\mathcal{L}_{1g} + \mathcal{L}_{1m})$ , where

$$\begin{aligned} \mathcal{L}_{1g} &= (n-1)\left\{-2\frac{\Lambda}{n-1}e^{2G} + [2F'R' + (n-2)R'^2] - \frac{\hat{\alpha}_2}{3}[4F'R'^3 + (n-4)R'^4]e^{-2G}\right. \\ &\quad \left. + \frac{\hat{\alpha}_3}{5}[6F'R'^5 + (n-6)R'^6]e^{-4G}\right\}e^{F-G+(n-1)R} \\ \mathcal{L}_{1m} &= \frac{1}{2}q^2(m^2 + H'^2e^{-2G})e^{-F+G+(n-1)R+2H}, \end{aligned} \quad (12)$$

where prime denotes the derivative with respect to  $r$ , and we define  $\hat{\alpha}_2 \equiv (n-2)(n-3)\alpha_2$  and  $\hat{\alpha}_3 \equiv (n-2)\dots(n-5)\alpha_3$  for convenience. The equations of motion following from this action may be written as:

$$\mathcal{L}_{1g} - \mathcal{L}_{1m} = \left\{ 2(n-1) \left( R' - \frac{2}{3}\hat{\alpha}_2 R'^3 e^{-2G} + \frac{3}{5}\hat{\alpha}_3 R'^5 e^{-4G} \right) e^{F-G+(n-1)R} \right\}', \quad (13)$$

$$\mathcal{L}_{1g} + \mathcal{L}_{1m} = \left\{ 2 \left[ F' + (n-2)R' - \frac{2}{3}\hat{\alpha}_2 (3F'R'^2 + (n-4)R'^3) e^{-2G} \right. \right. \quad (14)$$

$$\left. + \frac{3}{5}\hat{\alpha}_3 (5F'R'^4 + (n-6)R'^5) e^{-4G} \right] e^{F-G+(n-1)R} \right\}', \quad (15)$$

$$2\mathcal{L}_{1m} = \left\{ q^2 H' e^{-F-G+(n-1)R+2H} \right\}', \quad (16)$$

$$\begin{aligned} 0 = & \left[ 2F'R' + (n-2)R'^2 \right] - \hat{\alpha}_2 \left[ 4F'R'^3 + (n-4)R'^4 \right] e^{-2G} \\ & + \hat{\alpha}_3 \left[ 6F'R'^5 + (n-6)R'^6 \right] e^{-4G} + \frac{2\Lambda}{n-1} e^{2G} \\ & - \frac{q^2}{2(n-1)} (m^2 e^{2G} - H'^2) e^{2(H-F)}. \end{aligned} \quad (17)$$

Now subtracting the summation of Eqs. (13) and (16) from Eq. (15) one obtains:

$$\left\{ 2(F' - R') (1 - 2\hat{\alpha}_2 R'^2 e^{-2G} + 3\hat{\alpha}_3 R'^4 e^{-4G}) e^{F-G+(n-1)R} - q^2 H' e^{-F-G+(n-1)R+2H} \right\}' = 0,$$

which shows that

$$\begin{aligned} \mathcal{C}_0 = & 2(F' - R') (1 - 2\hat{\alpha}_2 R'^2 e^{-2G} + 3\hat{\alpha}_3 R'^4 e^{-4G}) e^{F-G+(n-1)R} - q^2 H' e^{-F-G+(n-1)R+2H} \\ = & \left\{ \left( 1 - 2\frac{\hat{\alpha}_2}{l^2} g + 3\frac{\hat{\alpha}_3}{l^4} g^2 \right) [r f' + 2(z-1)f] - q^2 (zh + rh')h \right\} \frac{r^{n+z-1}}{l^{z+1}} \left( \frac{f}{g} \right)^{1/2}, \end{aligned} \quad (18)$$

is conserved along the radial coordinate  $r$ . One may note that the action (12) and constant (18) reduce to those of Ref. [9] for  $n = 3$  and  $\hat{\alpha}_2 = \hat{\alpha}_3 = 0$ . This conserved quantity is associated with the shift

$$\begin{pmatrix} F(r) \\ R(r) \\ G(r) \\ H(r) \end{pmatrix} \rightarrow \begin{pmatrix} F(r) + \delta \\ R(r) - \frac{\delta}{n-1} \\ G(r) \\ H(r) + \delta \end{pmatrix}, \quad (19)$$

where  $\delta$  is a constant, and reduces to the diffeomorphism introduced in [9] for  $n = 3$  and  $\hat{\alpha}_2 = \hat{\alpha}_3 = 0$ .

We pause to remark that for  $z = 1$  with  $f(r) = g(r)$ , the constant (18) reduces to

$$\begin{aligned} \mathcal{C}_0 = & \frac{r^{n+1}}{l^2} \left( 1 - 2\frac{\hat{\alpha}_2}{l^2} f + 3\frac{\hat{\alpha}_3}{l^4} f^2 \right) f' \\ = & \frac{r^{n+1}}{l^2} \left( f - \frac{\hat{\alpha}_2}{l^2} f^2 + \frac{\hat{\alpha}_3}{l^4} f^3 \right)', \end{aligned}$$

which is known to be constant in third order Lovelock gravity and is proportional to the mass parameter of the spacetime [20].

### III. FINITE ACTION FOR LOVELOCK-LIFSHITZ SOLUTIONS

In Ref. [15], the authors define a finite action for Lifshitz theory in 4-dimensional Einstein gravity which satisfies  $\delta I = 0$  with appropriate boundary conditions by adding appropriate local counterterms to the action (4). In this section we generalize this action to the case of  $(n+1)$ -dimensional Lovelock gravity. As in the case of Einstein gravity, to preserve the diffeomorphism invariance of the action the counterterms rendering the action finite and yielding a well-defined variational principle should be covariant in the boundary fields. We consider  $I = I_{\text{bulk}} + I_{\text{bdy}}$ , where  $I_{\text{bulk}}$  is given in Eq. (4) and  $I_{\text{bdy}}$  is the sum of the boundary terms which are needed to have a well-defined variational principle and the counterterms which guarantees the finiteness of the action.  $I_{\text{bdy}}$ , for the case of zero curvature boundary which is our interest, may be written as

$$I_{\text{bdy}} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} \left\{ K - \frac{n-1}{l} + 2\alpha_2 \left( J - \frac{a(n)}{l^3} \right) + 3\alpha_3 \left( P - \frac{b(n)}{l^5} \right) + \frac{1}{2} f(A_\alpha A^\alpha) \right\} + I_{\text{deriv}}, \quad (20)$$

where the boundary  $\partial\mathcal{M}$  is the hypersurface at some constant  $r$ ,  $h_{\alpha\beta}$  is the induced metric,  $K$  is the trace of the extrinsic curvature,  $K_{\alpha\beta} = \nabla_{(\alpha} n_{\beta)}$  of the boundary (where the unit vector  $n^\mu$  is orthogonal to the boundary and outward-directed),  $a(n)$  and  $b(n)$  are two dimensionless constants depending on  $n$ , and  $J$  and  $P$  are the traces of [16]

$$J_{\alpha\beta} = \frac{1}{3} (2K K_{\alpha\gamma} K_{\beta}^{\gamma} + K_{\gamma\delta} K^{\gamma\delta} K_{\alpha\beta} - 2K_{\alpha\gamma} K^{\gamma\delta} K_{\delta\beta} - K^2 K_{\alpha\beta}) \quad (21)$$

$$P_{\alpha\beta} = \frac{1}{5} \left\{ [K^4 - 6K^2 K^{\gamma\delta} K_{\gamma\delta} + 8K K_{\gamma\delta} K_{\epsilon}^{\delta} K^{\epsilon\gamma} - 6K_{\gamma\delta} K^{\delta\epsilon} K_{\epsilon\psi} K^{\psi\gamma} + 3(K_{\gamma\delta} K^{\gamma\delta})^2] K_{\alpha\beta} - (4K^3 - 12K K_{\epsilon\delta} K^{\epsilon\delta} + 8K_{\delta\epsilon} K_{\psi}^{\epsilon} K^{\psi\delta}) K_{\alpha\gamma} K_{\beta}^{\gamma} - 24K K_{\alpha\gamma} K^{\gamma\delta} K_{\delta\epsilon} K_{\beta}^{\epsilon} + (12K^2 - 12K_{\epsilon\psi} K^{\epsilon\psi}) K_{\alpha\gamma} K^{\gamma\delta} K_{\delta\beta} + 24K_{\alpha\gamma} K^{\gamma\delta} K_{\delta\epsilon} K^{\epsilon\psi} K_{\beta\psi} \right\} \quad (22)$$

In Eq. (20),  $I_{\text{deriv}}$  is a collection of terms involving derivatives of the boundary fields, which could involve both the curvature tensor constructed from the boundary metric and covariant derivatives of  $A_\alpha$ . Since the boundary is flat and the fields are constants for 11, this term will not contribute to the on-shell value of the action for the pure Lifshitz solution or its first

variation around the Lifshitz background and therefore we ignore it throughout the paper. The matter part of the action (20) is the same as the matter part of the action in Einstein gravity. Thus, as in Ref. [15], an arbitrary function  $f(A^\alpha A_\alpha)$  is added to the action which is due to the fact that on the boundary  $A_\alpha A^\alpha = -q^2$  is constant for Lifshitz solutions.

The variation of the action about a solution of the equations of motion is just the boundary term,

$$\delta I = \frac{1}{16\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} \left\{ \Pi_{\alpha\beta} \delta h^{\alpha\beta} - n^\mu F_{\mu\nu} \delta A^\nu + f'(A_\alpha A^\alpha) (2A_\alpha \delta A^\alpha + A_\alpha A_\beta \delta h^{\alpha\beta}) - \frac{1}{2} f(A_\alpha A^\alpha) h_{\alpha\beta} \delta h^{\alpha\beta} \right\}, \quad (23)$$

where  $\Pi_{\alpha\beta} = \Pi_{\alpha\beta}^{(1)} + \Pi_{\alpha\beta}^{(2)} + \Pi_{\alpha\beta}^{(3)}$ , with

$$\begin{aligned} \Pi_{\alpha\beta}^{(1)} &= K_{\alpha\beta} - K h_{\alpha\beta} + \frac{n-1}{l} h_{\alpha\beta}, \\ \Pi_{\alpha\beta}^{(2)} &= 2\alpha_2 \left( 3J_{\alpha\beta} - J h_{\alpha\beta} + \frac{a(n)}{l^3} h_{\alpha\beta} \right), \\ \Pi_{\alpha\beta}^{(3)} &= 3\alpha_3 \left( 5P_{\alpha\beta} - P h_{\alpha\beta} + \frac{b(n)}{l^5} h_{\alpha\beta} \right). \end{aligned}$$

As in the case of Einstein gravity [15], one has for the Lovelock-Lifshitz spacetime (11)  $n^\mu F_{\mu\nu} \delta A^\nu = z q r^z / l^z \delta A^t$ ,

$$\Pi_{tt}^{(1)} = 0, \quad \Pi_{ij}^{(1)} = -\frac{z-1}{l} r^2 \delta_{ij},$$

and it is a matter of calculation to show that if one chooses

$$\begin{aligned} a(n) &= -(n-1)(n-2)(n-3), \\ b(n) &= (n-1)\dots(n-5), \end{aligned}$$

then one also has

$$\begin{aligned} \Pi_{tt}^{(2)} &= 0, \quad \Pi_{ij}^{(2)} = \frac{2(z-1)\hat{\alpha}_2}{l^3} r^2 \delta_{ij}, \\ \Pi_{tt}^{(3)} &= 0, \quad \Pi_{ij}^{(3)} = -\frac{3(z-1)\hat{\alpha}_3}{l^5} r^2 \delta_{ij}. \end{aligned}$$

That is,

$$\Pi_{tt} = 0, \quad \Pi_{ij} = \frac{(1-z)L^4}{l^5} r^2 \delta_{ij}. \quad (24)$$

Equation (24) shows that, as in the case of Einstein gravity [15], there are variations involving  $\delta h_{ij}$  and  $\delta A^t$  that we need to cancel. Using the same argument as in [15], the finite



action may be written as

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} (\mathcal{L}_g + \mathcal{L}_m) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} (K + 2\alpha_2 J + 3\alpha_3 P - \frac{(n-1)L^4}{l^5} - \frac{zq}{2l} \sqrt{-A_\alpha A^\alpha}) \quad (25)$$

It is remarkable that fixing a single coefficient suffices to cancel both divergences associated with  $\delta h^{ij}$  and  $\delta A^t$ . Note that when  $z = 1$ , then  $q = 0$  and these definitions reduce to the familiar AdS rules for third order Lovelock gravity [16]. If one defines

$$S_{\alpha\beta} = \frac{\sqrt{-h}}{16\pi} \left[ \Pi_{\alpha\beta} + \frac{zq}{2l} (-A_\alpha A^\alpha)^{-1/2} (A_\alpha A_\beta - A_\gamma A^\gamma h_{\alpha\beta}) \right], \quad (26)$$

$$S_\alpha = \frac{\sqrt{-h}}{16\pi} [n^\mu F_{\mu\alpha} + z\alpha (-A_\alpha A^\alpha)^{-1/2} A_\alpha], \quad (27)$$

then the general variation of the action is

$$\delta I = \int d^n x (S_{\alpha\beta} \delta h^{\alpha\beta} + S_\alpha \delta A^\alpha). \quad (28)$$

In the background (11), we have  $S_{\alpha\beta} = 0$ ,  $S_\alpha = 0$  due to cancellations between the different terms, and this action satisfies  $\delta I = 0$  for arbitrary variations around (11). Thus, we have a finite on-shell action which defines a well-defined variational principle for our background spacetime.

In the holographic renormalization programme for gauge-gravity duality in relativistic field theories, one computes the finite stress tensor after constructing a well-defined finite action, since it carries important physical information about the dual field theory. This job has been done for asymptotically AdS spacetimes [21, 22]. For asymptotically Lifshitz spacetimes, the dual field theory is nonrelativistic, so it will not have a covariant relativistic stress tensor. However one can define a stress tensor complex [15], consisting of the energy density  $\mathcal{E}$ , energy flux  $\mathcal{E}_i$ , momentum density  $\mathcal{P}_i$  and spatial stress tensor  $\mathcal{P}_{ij}$ , satisfying the conservation equations

$$\partial_t \mathcal{E} + \partial_i \mathcal{E}^i = 0, \quad \partial_t \mathcal{P}_j + \partial_i \mathcal{P}_j^i = 0, \quad (29)$$

where

$$\mathcal{E} = 2S^t_t - S^t A_t, \quad \mathcal{E}^i = 2S^i_t - S^i A_t, \quad (30)$$

and

$$\mathcal{P}_i = -2S^t_i + S^t A_i \quad \mathcal{P}_i^j = -2S^j_i + S^j A_i. \quad (31)$$

where in the Lovelock case  $S_{\alpha\beta}$  and  $S_\alpha$  are given by (26,27) respectively. Consequently the stress tensor complex (30) and (31) satisfies the conservation Eqs. (29).

#### IV. THE CONSTANT $\mathcal{C}_0$

In this section, we want to calculate the constant  $\mathcal{C}_0$ , which is conserved along the radial coordinate  $r$ . Since there is no exact Lovelock-Lifshitz solution (except under special circumstances), we calculate it at the horizon and at infinity. We will use this to relate the constant that appears in the expansion at  $r = \infty$  to the coefficients at the horizon.

##### A. $\mathcal{C}_0$ at the horizon

We begin by first reviewing the results found in [12] for the expansion near the horizon. Requiring that  $f(r)$  and  $g(r)$  go to zero linearly, that is

$$\begin{aligned} f(r) &= f_1 \{ (r - r_0) + f_2(r - r_0)^2 + f_3(r - r_0)^3 + f_4(r - r_0)^4 + \dots \}, \\ g(r) &= g_1(r - r_0) + g_2(r - r_0)^2 + g_3(r - r_0)^3 + g_4(r - r_0)^4 + \dots, \\ h(r) &= f_1^{1/2} \{ h_0 + h_1(r - r_0) + h_2(r - r_0)^2 + h_3(r - r_0)^3 + h_4(r - r_0)^4 + \dots \}, \end{aligned} \quad (32)$$

and inserting these expansions into the equations of motion arising from the action (4) for the metric (8) with the conditions (10) [Eqs (2.8)-(2.10) of Ref. [12]], and solving for the various coefficients, we find that  $h_0 = 0$ . This is consistent with the fact that the flux  $dA$  should go to a constant at the horizon.

By rescaling the time coordinate, we can adjust the constant  $f_1$  by an overall multiplicative factor (note the use of  $f_1^{1/2}$  in the expansion of  $h(r)$  as well, which is due to  $dt$  in the one-form  $A$ ). It is straightforward to find all the coefficients in terms of the two constants  $r_0$  and  $h_1$ .

Using the above expansion, we evaluate the constant  $\mathcal{C}_0$  (18) at  $r = r_0$ :

$$\mathcal{C}_0 = \frac{r_0^{n+z} \sqrt{f_1 g_1}}{l^{z+1}}. \quad (33)$$

This must be preserved along the flow in  $r$ .

## B. $\mathcal{C}_0$ at infinity

We now turn to the calculation of  $\mathcal{C}_0$  at large  $r$ . In order to do this, we investigate the behavior of the metric functions at large  $r$  by using straightforward perturbation theory:

$$\begin{aligned} f(r) &= 1 + \varepsilon f_1(r), \\ g(r) &= 1 + \varepsilon g_1(r), \\ h(r) &= 1 + \varepsilon h_1(r), \end{aligned}$$

and finding the field equations up to the first order in  $\varepsilon$ . We obtain

$$\begin{aligned} 0 &= 2r^2 h_1'' + 2(n+z)rh_1' + zr(g_1' - f_1') + 2(n-1)zg_1, \\ 0 &= 2(z-1)rh_1' + (n-1)rg_1' + [z(z-1) + n(n-1)]g_1 - (z-1)(n+z-1)(f_1 - 2h_1), \\ 0 &= 2(z-1)rh_1' + (n-1)rg_1' + [z(z-1) + n(n-1) + 2(n-1)(z-1)\mathcal{B}]g_1 \\ &\quad - (z-1)(z-n+1)(f_1 - 2h_1) \end{aligned} \tag{34}$$

where  $\mathcal{B} = (l^4 - 4\hat{\alpha}_2 l^2 + 9\hat{\alpha}_3)/L^4$ . Note that all the parameters of Lovelock gravity are in  $\mathcal{B}$ , with  $\mathcal{B} = 1$  in Einstein gravity.

The solution of Eqs. (34) is

$$\begin{aligned} h_1(r) &= -\frac{C_1}{r^{n+z-1}} - \frac{C_2}{r^{(n+z-1+\gamma)/2}} - \frac{C_3}{r^{(n+z-1-\gamma)/2}}, \\ f_1(r) &= -\frac{C_1 F_1}{r^{n+z-1}} - \frac{C_2 F_2}{r^{(n+z-1+\gamma)/2}} - \frac{C_3 F_3}{r^{(n+z-1-\gamma)/2}}, \\ g_1(r) &= -\frac{C_1 G_1}{r^{n+z-1}} - \frac{C_2 G_2}{r^{(n+z-1+\gamma)/2}} - \frac{C_3 G_3}{r^{(n+z-1-\gamma)/2}}, \end{aligned} \tag{35}$$

where

$$\begin{aligned}
\gamma &= \{(17 - 8\mathcal{B})z^2 - 2(3n + 9 - 8\mathcal{B})z + n^2 + 6n + 1 - 8\mathcal{B}\}^{1/2}, \\
F_1 &= 2(z - 1)(z - n + 1)\mathcal{K}^{-1}, \\
F_2 &= (\mathcal{F}_1 - \mathcal{F}_2)\{8z\mathcal{K}[(z - 1)\mathcal{B} + 2n + z - 3]\}^{-1}, \\
F_3 &= (\mathcal{F}_1 + \mathcal{F}_2)\{8z\mathcal{K}[(z - 1)\mathcal{B} + 2n + z - 3]\}^{-1}, \\
G_1 &= 2(z - 1)(n + z - 1)\mathcal{K}^{-1}, \\
G_2 &= (\mathcal{G}_1 + \mathcal{G}_2)\{8z\mathcal{K}[(z - 1)\mathcal{B} + 2n + z - 3]\}^{-1}, \\
G_3 &= (\mathcal{G}_1 - \mathcal{G}_2)\{8z\mathcal{K}[(z - 1)\mathcal{B} + 2n + z - 3]\}^{-1}, \\
\mathcal{K} &= (z - 1)(n + z - 1)\mathcal{B} + z(z - 1) + n(n - 1), \\
\mathcal{F}_1 &= 8(z - 1)[(z - 1)(n + z - 1)\mathcal{B} + z(z - 1) + n(n - 1)] \\
&\quad \times [(z - 1)(n + 3z - 3)\mathcal{B} - 2z^2 + (n + 3)z + n(n - 2) - 1], \\
\mathcal{F}_2 &= \gamma[n - 1 + (z - 1)\mathcal{B}]\left\{8(1 + \mathcal{B})(z - 1)^3 \right. \\
&\quad \left. + (17n - 9 + 8\mathcal{B})(z - 1)^2 + 2(n + 8)(n - 1)(z - 1) + n^2(n - 1) - (n - 1)\gamma^2\right\}, \\
\mathcal{G}_1 &= 8(z - 1)[2(z - 1)\mathcal{B} - 3z + 3n - 1][(z - 1)(n + z - 1)\mathcal{B} + z(z - 1) + n(n - 1)], \\
\mathcal{G}_2 &= \left\{8(1 + \mathcal{B})(z - 1)^3 + (17n - 9 + 8\mathcal{B})(z - 1)^2 \right. \\
&\quad \left. + 2(n + 8)(n - 1)(z - 1) + n^2(n - 1)\right\}\gamma - (n - 1)\gamma^3.
\end{aligned}$$

The case of  $z = n - 1$  in Einstein gravity needs special consideration. In this case  $\gamma = 2z$ , and therefore the second term in Eqs. (35) is exactly the same as the first term. Also, one should choose  $C_3 = 0$  in order to have suitable asymptotically behavior for the functions. It is a matter of calculation to show that the solution of Eqs. (34) for  $z = n - 1$ , together with the constraint that these functions should go to zero for large  $r$ , may be written as

$$\begin{aligned}
h_1(r) &= -\frac{C_1 + C_2 \ln r}{r^{2(n-1)}}, \\
f_1(r) &= -\frac{3n - 4}{(n - 1)(2n - 3)} \frac{C_2}{r^{2(n-1)}}, \\
g_1(r) &= -\frac{2(n - 2)(C_1 + C_2 \ln r)}{(2n - 3)r^{2(n-1)}} - \frac{(n^2 - 2)C_2}{(n - 1)(2n - 3)^2 r^{2(n-1)}}, \tag{36}
\end{aligned}$$

where  $C_1$  and  $C_2$  are integration constants.

Now we want to calculate the conserved quantity  $\mathcal{C}_0$ . We restrict ourselves to the case that  $\gamma \geq n + z - 1$ , which for Einstein gravity means that  $z \geq n - 1$ . However in Lovelock

gravity, this condition holds provided  $z \geq (n - \mathcal{B})/(2 - \mathcal{B})$ , where  $\mathcal{B}$  depends on the Lovelock coefficients. Note that this condition does not hold for  $\mathcal{B} \geq 2$ . If  $z \geq (n - \mathcal{B})/(2 - \mathcal{B})$  then  $C_3 = 0$ , since at large  $r$  the functions  $f_1(r)$ ,  $g_1(r)$  and  $h_1(r)$  should go to zero as  $r$  goes to infinity, and the contribution in  $\mathcal{C}_0$  from  $C_2$  is zero. Thus, one has only the first terms in the expansions (35) and the constant  $\mathcal{C}_0$  can be obtained as

$$\mathcal{C}_0 = \frac{2(z-1)(z+n-1)^2\{(z-n+1)l^4 + 2(n-2)\hat{\alpha}_2 l^2 - 3(n+z-3)\hat{\alpha}_3\}}{z l^{z+5} \mathcal{K}} C_1. \quad (37)$$

We also find that the terms due to the second order perturbation do not contribute to the conserved quantity (37).

The above constant (37) in Einstein gravity for  $z = n - 1$  is zero. In this case we must use the expansion (36) to obtain the conserved quantity  $\mathcal{C}_0$ , which gives

$$\mathcal{C}_0 = \frac{4(n-1)}{(2n-3)l^n} C_2, \quad (z = n - 1, \text{ Einstein}) \quad (38)$$

## V. BLACK BRANE THERMODYNAMICS

The entropy of a black hole in Lovelock gravity is [23]

$$S = \frac{1}{4} \sum_{k=1}^p k \alpha_k \int d^{n-1}x \sqrt{\tilde{g}} \tilde{\mathcal{L}}_{k-1}, \quad (39)$$

where the integration is done on the  $(n-1)$ -dimensional spacelike hypersurface of the Killing horizon with induced metric  $\tilde{g}_{\mu\nu}$  (whose determinant is  $\tilde{g}$ ), and  $\tilde{\mathcal{L}}_k$  is the  $k$ th order Lovelock Lagrangian of  $\tilde{g}_{\mu\nu}$ . The entropy of a black brane per unit volume of the horizon in third order Lovelock gravity is

$$S = \frac{1}{4} r_0^{n-1}. \quad (40)$$

The temperature of the event horizon can be obtained by using the expansion (32), yielding

$$T = \frac{r_0^{z+1} \sqrt{f_1 g_1}}{4\pi l^{z+1}}. \quad (41)$$

Using the value of  $\mathcal{C}_0$  at horizon (33) and Eqs. (40) and (41), we find

$$\mathcal{C}_0 = 16\pi T S. \quad (42)$$

Alternatively, one can calculate the energy density of the black brane by using Eqs. (26), (27) and (30). We find that

$$\mathcal{E} = \frac{(n-1)(z-1)(z+n-1)\{(z-n+1)l^4 + 2(n-2)\hat{\alpha}_2 l^2 - 3(n+z-3)\hat{\alpha}_3\}}{8\pi z l^{z+5} \mathcal{K}} C_1. \quad (43)$$

For  $z = n - 1$  the above expression for the energy density in Einstein gravity vanishes and we must use the expansion (36) to find the energy density in Einstein gravity with  $z = n - 1$ . This gives

$$\mathcal{E} = \frac{(n-1)}{8\pi(2n-3)l^n} C_2. \quad (44)$$

Dividing Eq. (43) by Eq. (37) (or, for  $z = n - 1$ , (44) by (38)) and using Eq. (42) we find

$$\mathcal{E} = \frac{n-1}{n+z-1} TS. \quad (45)$$

which is valid for all  $z$ . Note that both the constant  $\mathcal{C}_0$  and the energy density  $\mathcal{E}$  are different from their counterparts in Einstein gravity since the coefficient of  $r^{-(n+z-1)}$  at large  $r$  for the functions  $f(r)$ ,  $g(r)$  and  $h(r)$  differs. However, as one can see from Eq. (45),  $\mathcal{E}(T, S)$  is the same in both Einstein [17] and Lovelock gravity.

For the case of  $z = 1$ , for which the field equations have exact solution, the energy density, entropy density and temperature are [16]

$$\begin{aligned} \mathcal{E} &= \frac{(n-1)m}{16\pi} = \frac{(n-1)r_0^n}{16\pi l^2}, \\ T &= \frac{nr_0}{4\pi l^2}, \quad S = \frac{1}{4}r_0^{n-1}, \end{aligned}$$

and therefore  $\mathcal{E} = (n-1)ST/n$ , consistent with Eq. (45) for  $z = 1$ .

Using the first law of thermodynamics  $d\mathcal{E} = TdS$  with the relation (45) for the energy density, one obtains

$$\log T = \frac{z}{n-1} \log S + \Gamma, \quad (46)$$

where  $\Gamma$  is the integration constant that depends on the Lovelock coefficients,  $z$  and the dimension of spacetime. It can be found numerically. This result is consistent with our previous numerical solution [12], which shows that the slope of  $\log T$  versus  $\log S$  are the same for Einstein and Lovelock gravity, while  $\Gamma$  is different. Furthermore, by use of Eqs. (40) and (46), the temperature is proportional to  $r_0^z$ .

## VI. CONCLUSIONS

In this paper, we introduced the finite action of third order Lovelock gravity in the presence of a massive vector field with a flat boundary. Indeed, we generalized the counterterm method introduced in [16] for asymptotically AdS black branes to the case of asymptotic

Lifshitz black branes. We also defined the finite stress tensor complex, and computed the energy density of the Lovelock-Lifshitz black branes. We then used the field equations to find a conserved quantity along the  $r$  coordinate. This constant, which is the generalization of the constant introduced in [9], has the role of connecting the metric parameters at the horizon and at infinity. We used these generalizations to investigate the thermodynamics of Lovelock-Lifshitz black branes introduced in [12]. We found that the relationship between the energy density, temperature, and entropy density is unchanged from Einsteinian gravity, even though the subleading large- $r$  behavior of Lovelock-Lifshitz black branes is different from the Einsteinian Lifshitz solutions. We made use of the first law of thermodynamics to obtain the relationship between entropy and temperature. We found this is also the same as the Einsteinian case, apart from a constant of integration that depends on the parameters in the Lovelock action. These results are consistent with the numerical analysis of Lovelock-Lifshitz black branes investigated in [12].

Our counterterm method can be applied only to the case of  $k = 0$  solutions. It would be interesting to generalize this method to the  $k = \pm 1$  cases. Its generalization to quasi-topological gravity with cubic-curvature terms that are not supersymmetric and therefore is different from the third order Lovelock gravity in the context of holography [14] would also be of interest. It is known that the third order Lovelock term does not contribute to the three-point functions of gravitons in flat space [24], while the corresponding contribution in the AdS background does not vanish [13]. Employing our action to calculate two and three-point functions in the pure Lifshitz background, and considering their differences with respect to the case of asymptotic flat and AdS solutions is another topic worth investigating.

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